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New Method to Evaluating Exact and Traveling Wave Solutions for Non Linear Evolution Equations

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The extended $\exp(-\varphi(\xi))$-expansion method is used as the first time to investigate the wave solution of a nonlinear dynamical system in a new double-Chain model of DNA, the nonlinear Burger equation with power law non linearity and the perturbed nonlinear Schrodinger equation with Kerr law non linearity. The proposed method give a wide range for the solutions and it also can be used for many other nonlinear evolution equations.

Keywords: Extended $\exp(-\varphi(\xi))$-Expansion Method, Dynamical System in a New Double-Chain Model of DNA, The Nonlinear Burger Equation with Power Law Non Linearity, The Perturbed Nonlinear Schrodinger Equation with Kerr Law Non Linearity, Traveling Wave Solutions, Solitary Wave Solutions.

RESEARCH ARTICLE

1. INTRODUCTION

The nonlinear partial differential equations of mathematical physics are major subjects in physical science.[1] No one can deny the important role which played by the nonlinear partial differential equations in the description of many and a wide variety of phenomena not only in physical phenomena, but also in plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena. So that, during the past five decades, a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations, for example, extended Jacobian Elliptic Function Expansion Method,[2] the modified simple equation method,[3] the tanh method,[4] extended tanh-method,[5–7] sine-cosine method,[8–10] homogeneous balance method,[11,12] F-expansion method,[13–15] exp-function method,[16,17] trigonometric function series method,[18] (G'/G)-expansion method,[19–22] Jacobi elliptic function method.[23–26] The $\exp(-\varphi(\xi))$-expansion method[27–29] and so on.

In this article we propose a new method to get the exact traveling wave solutions and the solitary wave solutions of dynamical system in a new double-Chain model of DNA, the nonlinear Burger equation with power law non linearity and the perturbed nonlinear Schrodinger equation with Kerr law non linearity by using the extended $\exp(-\varphi(\xi))$-expansion method. Where these equations play an important role in biology and mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of the extended $\exp(-\varphi(\xi))$-expansion method. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. DESCRIPTION OF METHOD

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0 \tag{1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following,we give the main steps of this method.

Step 1. We use the wave transformation

$$u(x,t) = u(\xi), \quad \xi = x - ct \tag{2}$$

where c is a positive constant, to reduce Eq. (1) to the following ODE:

$$P(u, u', u'', u''', \ldots) = 0 \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = d/d\xi'$.

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $\exp(-\varphi(\xi))$ as follows

$$u(\xi) = \sum_{i=-m}^{m} a_i(\exp(-\varphi(\xi)))^i \tag{4}$$

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Since a_m $(0 \leq m \leq n)$ are constants to be determined, such that a_m or $a_{-m} \neq 0$.

The positive integer m can be determined by considering the homogenous balance between the highest order derivatives and nonlinear terms appearing in Eq. (3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = m$, which gives rise to degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n+q, \quad D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q)$$

Therefore, we can find the value of m in Eq. (3), where $\varphi = \varphi(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda \tag{5}$$

the solutions of ODE (3) are
when $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1)) - \lambda}{2\mu}\right) \tag{6}$$

and

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1)) - \lambda}{2\mu}\right) \tag{7}$$

when $\lambda^2 - 4\mu > 0$, $\mu = 0$,

$$\varphi(\xi) = -\ln\left(\frac{\lambda}{\exp(\lambda(\xi+C_1)) - 1}\right) \tag{8}$$

when $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$\varphi(\xi) = \ln\left(-\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)}\right) \tag{9}$$

when $\lambda^2 - 4\mu = 0$, $\mu = 0$, $\lambda = 0$,

$$\varphi(\xi) = \ln(\xi + C_1) \tag{10}$$

when $\lambda^2 - 4\mu < 0$,

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}{2\mu}\right) \tag{11}$$

and

$$\varphi(\xi) = \ln\left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}{2\mu}\right) \tag{12}$$

where $a_m, \ldots, \lambda, \mu$ are constants to be determined later.
Step 3. After we determine the index parameter m, we substitute Eq. (4) along Eq. (5) into Eq. (3) and collecting all the terms of the same power $\exp(-m\varphi(\xi))$, $m = 0, 1, 2, 3, \ldots$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i.
Step 4. Substituting these values and the solutions of Eq. (5) into Eq. (3) we obtain the exact solutions of Eq. (3).

3. APPLICATION

3.1. Example 1: Dynamical System in a New Double-Chain Model of DNA

An attractive nonlinear model for the nonlinear science in the deoxyribonucleic acid (DNA). The dynamics of DNA molecules is one of the most fascinating problems of modern biophysics because it is at the basis of life. The DNA structure has been studied during last decades. The investigation of DNA dynamics has successfully predicted the appearance of important nonlinear structures. It has been shown that the non linearity is responsible for forming localized waves. These localized waves are interesting because they have the capability to transport energy without dissipation.[30–38] In Ref. [37, 38], it is given that a new double-chain model of DNA consists of two long elastic homogeneous strands which represent two poly nucleotide chains of the DNA molecule, connected with each other by an elastic membrane representing the hydrogen bonds between the base pair of the two chains. Under some appropriate approximation, the new double-chain model of DNA can be described by the following two general nonlinear dynamical system:

$$u_{tt} - c_1^2 u_{xx} = \lambda_1 u + \gamma_1 uv + \mu_1 u^3 + \beta_1 uv^2 \tag{13}$$

$$v_{tt} - c_2^2 v_{xx} = \lambda_2 v + \gamma_2 u^2 + \mu_2 u^2 v + \beta_2 v^3 + c_0 \tag{14}$$

where

$$\begin{gathered} c_1 = \pm\frac{Y}{\rho}; \quad c_2 = \pm\frac{F}{\rho}; \quad \lambda_1 = \frac{-2\mu}{\rho\sigma h}(c - l_0) \\ \lambda_2 = \frac{-2\mu}{\rho\sigma}; \quad \gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2} \\ \mu_1 = \mu_2 = \frac{-2\mu l_0}{\rho\sigma h^3}, \quad \beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3} \\ c_0 = \frac{\sqrt{2}\mu(h-l_0)}{\rho\sigma} \end{gathered} \tag{15}$$

where ρ, σ, Y and F denote respectively the mass density, the area of transverse cross-section, the Young's modulus and tension density of each strand; μ is the rigidity of the elastic membrane; h is the distance between the two strands, and l_0 is the height of the membrane in the equilibrium positive. In Eqs. (13) and (14), u is the difference of the longitudinal displacements of the bottom and top strands, while v is the difference of the transverse displacement of the bottom and top strands.

We first introduce the transformation

$$v = au + b \tag{16}$$

where a and b are constants, to reduce Eqs. (13) and (14) to the following system of equations:

$$\begin{aligned} u_{tt} - c_1^2 u_{xx} &= u^3(\mu_1 + \beta_1 a^2) + u^2(2\beta_1 ab + a\gamma_1) \\ &\quad + u(\lambda_1 + b\gamma_1 + \beta_1 b^2) \end{aligned} \tag{17}$$

and

$$u_{tt}-c_2^2u_{xx}=u^3(\mu_2+\beta_2a^2)+u^2\left(\frac{\gamma_2}{a}+\frac{\mu_2b}{a}+3\beta_2ab\right)+u(\lambda_2+3\beta_2b^2)+\frac{\lambda_2b}{a}+\frac{\beta_2b^3}{a}+\frac{c_0}{a} \quad (18)$$

Comparing Eqs. (17) and (18) and using (16) we deduce that $b=h/\sqrt{2}$ and $F=Y$. Now Eqs. (17) and (18) can be written as

$$u_{tt}-c_1^2u_{xx}-Au^3-Bu^2-Cu=0 \quad (19)$$

where

$$A=\frac{\alpha}{h^3}(-2+4a^2);\quad B=\frac{6\sqrt{2}a\alpha}{h^2}$$
$$C=\left(\frac{-2\alpha}{l_0}+\frac{6\alpha}{h}\right);\quad \alpha=\frac{\mu l_0}{\rho\sigma};\quad c_1^2=\frac{Y}{\rho} \quad (20)$$

The wave transformation $u(x,t)=u(\xi)$, $\xi=kx+\omega t$, reduce Eq. (19) to the following ODE:

$$(\omega^2-k^2c_1^2)u''-Au^3-Bu^2-Cu=0 \quad (21)$$

where $\omega^2-k^2c_1^2\neq 0$. Balancing u'' and u^3 yields, $N+2=3N\to N=1$. Consequently, we have the formal solution:

$$u(\xi)=a_{-1}\exp(\varphi(\xi))+a_0+a_1\exp(-\varphi(\xi)) \quad (22)$$

Substituting Eq. (22) and its derivative into Eq. (21) and collecting all term with the same power of $[\exp(-3\varphi(\xi)), \exp(-2\varphi(\xi)),\ldots,\exp(+3\varphi(\xi))]$ we obtained:

$$(\omega^2-k^2c_1^2)\mu^2a_{-1}-Aa_{-1}^3=0 \quad (23)$$

$$3(\omega^2-k^2c_1^2)\mu\lambda a_{-1}-3Aa_{-1}^2a_0-Ba_{-1}^2=0 \quad (24)$$

$$(\omega^2-k^2c_1^2)(2a_{-1}\mu+\lambda^2a_{-1})-A(3a_{-1}^2a_1+3a_{-1}a_0^2)-2Ba_{-1}a_0-Ca_{-1}=0 \quad (25)$$

$$(\omega^2-k^2c_1^2)(\lambda a_{-1}+\mu\lambda a_1)-A(6a_{-1}a_0a_1+a_0^3)-B(2a_{-1}a_1+a_0^2)-Ca_0=0 \quad (26)$$

$$(\omega^2-k^2c_1^2)(2\mu a_1+\lambda^2a_1)-A(3a_{-1}a_1^2+3a_0^2a_1)-2Ba_0a_1-Ca_1=0 \quad (27)$$

$$3(\omega^2-k^2c_1^2)\lambda a_1-3Aa_0a_1^2-Ba_1^2=0 \quad (28)$$

$$2(\omega^2-k^2c_1^2)a_1-Aa_1^3=0 \quad (29)$$

Solving above system by using maple 16, we get the following solutions:
Sol. 1.

$$A=-8\frac{a_{-1}^2(-\omega^2+k^2c_1^2)}{a_0^4},\quad B=24\frac{a_{-1}^2(-\omega^2+k^2c_1^2)}{a_0^3}$$
$$C=-16\frac{a_{-1}^2(-\omega^2+k^2c_1^2)}{a_0^2},\quad \mu=2\frac{a_{-1}^2}{a_0^2}$$
$$\lambda=0,\quad a_{-1}=a_{-1},\quad a_0=a_0,\quad a_1=\frac{a_0^2}{2a_{-1}}$$

Sol. 2.

$$A=-32\frac{a_{-1}^2(-\omega^2+k^2c_1^2)}{a_0^4},\quad B=96\frac{a_{-1}^2(-\omega^2+k^2c_1^2)}{a_0^3}$$
$$C=-64\frac{a_{-1}^2(-\omega^2+k^2c_1^2)}{a_0^2},\quad \mu=-4\frac{a_{-1}^2}{a_0^2}$$
$$\lambda=0,\quad a_{-1}=a_{-1},\quad a_0=a_0,\quad a_1=\frac{a_0^2}{4a_{-1}}$$

Sol. 3.

$$A=-2\frac{\mu^2(-\omega^2+k^2c_1^2)}{a_{-1}^2}$$
$$B=3\frac{\mu(-\mu a_0^2\omega^2+\omega^2a_{-1}^2+\mu a_0^2k^2c_1^2-k^2c_1^2a_{-1}^2)}{a_{-1}^2a_0}$$
$$C=-(-\mu^2a_0^4\omega^2+2\omega^2\mu a_{-1}^2a_0^2+\mu^2a_0^4k^2c_1^2-2k^2c_1^2\mu a_{-1}^2a_0^2-a_{-1}^4\omega^2+a_{-1}^4k^2c_1^2)\times(a_0^2a_{-1}^2)^{-1}$$
$$\lambda=\frac{\mu a_0^2+a_{-1}^2}{a_{-1}a_0},\quad a_{-1}=a_{-1},\quad a_0=a_0,\quad a_1=0$$

Sol. 4.

$$A=-2\frac{-\omega^2+k^2c_1^2}{a_1^2}$$
$$B=-3\frac{-a_1\lambda\omega^2+a_1\lambda k^2c_1^2+2a_0\omega^2-2a_0k^2c_1^2}{a_1^2}$$
$$C=-(5a_0a_1\lambda\omega^2-5a_0a_1\lambda k^2c_1^2-4a_0^2\omega^2+4a_0^2k^2c_1^2-\omega^2\lambda^2a_1^2+k^2c_1^2\lambda^2a_1^2)\times(a_1^2)^{-1}$$
$$\mu=\frac{-a_0(a_1\lambda-2a_0)}{4a_1^2},\quad a_{-1}=\frac{-a_0(a_1\lambda-2a_0)}{4a_1}$$
$$a_0=a_0,\quad a_1=a_1$$

Sol. 5.

$$\mu=\frac{a_{-1}}{a_1},\quad B=3\frac{\lambda(-\omega^2+k^2c_1^2)}{a_1}$$
$$C=-\frac{-\omega^2\lambda^2a_1+k^2c_1^2\lambda^2a_1+4\omega^2a_{-1}-4k^2c_1^2a_{-1}}{a_1}$$
$$A=-2\frac{-\omega^2+k^2c_1^2}{a_1^2},\quad a_{-1}=a_{-1},\quad a_0=a_1\lambda,\quad a_1=a_1$$

For Sol. 1.

$$u(\xi)=a_{-1}\exp(\varphi(\xi))+a_0+\frac{a_0^2}{2a_{-1}}\exp(-\varphi(\xi)) \quad (30)$$

Let us discuss the only this case where the other cases do not satisfy the conditions of solutions for (5):

$$u(\xi)=a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0+\frac{a_0^2}{2a_{-1}}\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \quad (31)$$

and

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + \frac{a_0^2}{2a_{-1}}\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \quad (32)$$

For Sol. 2.

$$u(\xi) = a_{-1}\exp(\varphi(\xi))+a_0+\frac{a_0^2}{2a_{-1}}\exp(-\varphi(\xi)) \quad (33)$$

Let us discuss the only this case where the other cases do not satisfy the conditions of solutions for (5):
when $\lambda^2-4\mu>0,\ \mu\neq 0$,

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + \frac{a_0^2}{2a_{-1}}\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \quad (34)$$

and

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + \frac{a_0^2}{2a_{-1}}\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \quad (35)$$

For Sol. 3.

$$u(\xi) = a_{-1}\exp(\varphi(\xi))+a_0 \quad (36)$$

Let us discuss the only these three cases where the other cases do not satisfy the conditions of solutions for (5):
when $\lambda^2-4\mu>0,\ \mu\neq 0$,

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 \quad (37)$$

and

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 \quad (38)$$

when $\lambda^2-4\mu=0,\ \mu\neq 0,\ \lambda\neq 0$,

$$u(\xi) = a_{-1}\left(-\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)}\right)+a_0 \quad (39)$$

when $\lambda^2-4\mu<0$,

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 \quad (40)$$

and

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 \quad (41)$$

For Sol. 4.

$$u(\xi) = \frac{-a_0(a_1\lambda-2a_0)}{4a_1}\exp(\varphi(\xi))+a_0 + a_1\exp(-\varphi(\xi)) \quad (42)$$

Let us discuss the only these three cases where the other cases do not satisfy the conditions of solutions for (5):
when $\lambda^2-4\mu>0,\ \mu\neq 0$,

$$u(\xi) = \frac{-a_0(a_1\lambda-2a_0)}{4a_1} \times\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + a_1\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \quad (43)$$

and

$$u(\xi) = \frac{-a_0(a_1\lambda-2a_0)}{4a_1} \times\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + a_1\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \quad (44)$$

when $\lambda^2-4\mu=0,\ \mu\neq 0,\ \lambda\neq 0$,

$$u(\xi) = \frac{-a_0(a_1\lambda-2a_0)}{4a_1}\left(-\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)}\right)+a_0 + a_1\left(-\frac{\lambda^2(\xi+C_1)}{2(\lambda(\xi+C_1)+2)}\right) \quad (45)$$

when $\lambda^2-4\mu<0$,

$$u(\xi) = \frac{-a_0(a_1\lambda-2a_0)}{4a_1} \times \left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \quad (46)$$

and

$$u(\xi) = \frac{-a_0(a_1\lambda-2a_0)}{4a_1} \times \left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_0 + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \quad (47)$$

For Sol. 5.

$$u(\xi)=a_{-1}\exp(\varphi(\xi))+a_1\lambda+a_1\exp(-\varphi(\xi)) \quad (48)$$

Let us discuss the only these three cases where the other cases do not satisfy the conditions of solutions for (5): when $\lambda^2-4\mu>0$, $\mu\neq 0$,

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_1\lambda + a_1\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \quad (49)$$

and

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_1\lambda + a_1\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \quad (50)$$

when $\lambda^2-4\mu>0$, $\mu=0$,

$$u(\xi)=a_{-1}\left(\frac{\exp(\lambda(\xi+C_1))-1}{\lambda}\right)+a_1\lambda + a_1\left(\frac{\lambda}{\exp(\lambda(\xi+C_1))-1}\right) \quad (51)$$

when $\lambda^2-4\mu=0$, $\mu\neq 0$, $\lambda\neq 0$,

$$u(\xi)=a_{-1}\left(-\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)}\right)+a_1\lambda + a_1\left(-\frac{\lambda^2(\xi+C_1)}{2(\lambda(\xi+C_1)+2)}\right) \quad (52)$$

when $\lambda^2-4\mu<0$,

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_1\lambda + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \quad (53)$$

and

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+a_1\lambda + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \quad (54)$$

3.2. Example 2. The Nonlinear Burger Equation with Power Law Non Linearity

This equation is well known[39] and has the form:

$$v_t+a(v^n)_x+bv_{xx}=0, \quad n>1 \quad (55)$$

where a and b are nonzero constants. The solutions of Eq. (55) have been discussed, the exact solitary wave solutions, the periodic solutions and the rational function solution are obtained in Ref. [39] by means of the extended (G'/G)-expansion method. Let us now solve Eq. (55) using the extended $\exp(-\varphi(\xi))$-expansion method. To this end, we use the wave transformation (14) to reduce Eq. (55) to the ODE and integrating the equation with zero constant of integration:

$$-cv+av^n+bv'=0 \quad (56)$$

Balancing v' with v^n yields $m=1/n-1$, $n>1$. Using the transformation

$$v=u^{1/(n-1)} \quad (57)$$

to reduce Eq. (56) to the following equation

$$-c(n-1)u+a(n-1)u^2+bu'=0 \quad (58)$$

where u is a new function of ξ. Balancing u' with u^2 yields $m=1$. Consequently, Eq. (58) has the same formal solution of (21). Substituting Eq. (22) and its derivative into Eq. (58) and collecting all term with the same power of $[\exp(-2\varphi(\xi)), \exp(-1\varphi(\xi)), \ldots, \exp(+2\varphi(\xi))]$ we obtained:

$$a(n-1)a_{-1}^2 + b\mu a_{-1} = 0 \tag{59}$$

$$-c(n-1)a_{-1} + 2a(n-1)a_{-1}a_0 + b\lambda a_{-1} = 0 \tag{60}$$

$$-c(n-1)a_0 + a(n-1)(2a_{-1}a_1 + a_0^2) + b(a_{-1} - \mu a_1) = 0 \tag{61}$$

$$-c(n-1)a_1 + 2a(n-1)a_0a_1 - b\lambda a_1 = 0 \tag{62}$$

$$a(n-1)a_1^2 - ba_1 = 0 \tag{63}$$

Solving above system by using maple 16, we get the following solutions:
Sol. 1.

$$b = aa_1n - aa_1, \quad c = 2aa_0 - \lambda aa_1, \quad \mu = \frac{a_0(a_1\lambda - a_0)^2}{a_1}$$

$$a_{-1} = 0, \quad a_0 = a_0, \quad a_1 = a_1$$

Sol. 2.

$$b = -\frac{a(n-1)a_{-1}}{\mu}, \quad c = \frac{a(-a_{-1}^2 + \mu a_0^2)}{a_0\mu}$$

$$\lambda = \frac{\mu a_0^2 + a_{-1}^2}{a_{-1}a_0}, \quad a_{-1} = a_{-1}, \quad a_0 = a_0, \quad a_1 = 0$$

Sol. 3.

$$b = aa_1n - aa_1, \quad \mu = \frac{-c^2}{16a^2a_1^2}, \quad \lambda = 0$$

$$a_{-1} = \frac{c^2}{16a^2a_1}, \quad a_0 = \frac{c}{2a}, \quad a_1 = a_1$$

For Sol. 1.

$$u(\xi) = a_0 + a_1\exp(-\varphi(\xi)) \tag{64}$$

Let us investigate these cases only where the other cases do not satisfy the conditions of solutions for (5):
when $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$u(\xi) = a_0 + a_1\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1)) - \lambda}\right) \tag{65}$$

and

$$u(\xi) = a_0 + a_1\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1)) - \lambda}\right) \tag{66}$$

when $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$u(\xi) = a_0 + a_1\left(-\frac{\lambda^2(\xi + C_1)}{2(\lambda(\xi + C_1) + 2)}\right) \tag{67}$$

when $\lambda^2 - 4\mu < 0$,

$$u(\xi) = a_0 + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}\right) \tag{68}$$

and

$$u(\xi) = a_0 + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}\right) \tag{69}$$

For Sol. 2.

$$u(\xi) = a_{-1}\exp(\varphi(\xi)) + a_0 \tag{70}$$

Let us investigate these cases only where the other cases do not satisfy the conditions of solutions for (5):
when $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1)) - \lambda}{2\mu}\right) + a_0 \tag{71}$$

and

$$u(\xi) = a_{-1}\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1)) - \lambda}{2\mu}\right) + a_0 \tag{72}$$

when $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$u(\xi) = a_{-1}\left(-\frac{2(\lambda(\xi + C_1) + 2)}{\lambda^2(\xi + C_1)}\right) + a_0 \tag{73}$$

when $\lambda^2 - 4\mu < 0$,

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}{2\mu}\right) + a_0 \tag{74}$$

and

$$u(\xi) = a_{-1}\left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}{2\mu}\right) + a_0 \tag{75}$$

For Sol. 3.

$$u(\xi) = \frac{c^2}{16a^2a_1}\exp(\varphi(\xi)) + \frac{c}{2a} + a_1\exp(-\varphi(\xi)) \tag{76}$$

Let us investigate the only this case where the other cases do not satisfy the conditions of solutions for (5):
when $\lambda^2 - 4\mu < 0$,

$$u(\xi) = \frac{c^2}{16a^2a_1} \times \left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}{2\mu}\right) + \frac{c}{2a} + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}\right) \tag{77}$$

and

$$u(\xi) = \frac{c^2}{16a^2a_1} \times \left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}{2\mu}\right) + \frac{c}{2a} + a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1)) - \lambda}\right) \tag{78}$$

3.3. Example 3. The Perturbed Nonlinear Schrodinger Equation with Kerr Law Non Linearity

This equation is well-known[40,41] and has the form:

$$iu_t + u_{xx} + \alpha|u|^2u + i\{\gamma_1 u_{xxx} + \gamma_2|u|^2u_x + \gamma_3(|u|^2)_x u\} = 0 \tag{79}$$

where α, γ_1, γ_2, γ_3 are constants such that γ_1 is the third order dispersion, γ_2 is the nonlinear dispersion, while γ_3 is also a version of nonlinear dispersion.[42,43] Equation (79) describes the propagation of optical solitons in nonlinear optical fibers that exhibits a Kerr law non linearity. Equation (79) has been discussed in Ref. [41] using the first integral method and in Ref. [40] using the modified mapping method and its extended. Let us now solve Eq. (79) using the extended $\exp(-\varphi(\xi))$-expansion method. To this end we seek its traveling wave solution of the form Refs. [40, 41]:

$$u(x,t) = \phi(\xi)\exp[i(kx - \Omega t)], \quad \xi = x - ct \tag{80}$$

where k, Ω and c are constants, while $i = \sqrt{-1}$. Substituting (80) into Eq. (79) and equating the real and imaginary parts to zero, we have

$$\gamma_1\phi''' + (2k - c - 3\gamma_1k^2)\phi' + (\gamma_2 + 2\gamma_3)\phi^2\phi' = 0 \tag{81}$$

and

$$(1-3\gamma_1k)\phi'' + (\Omega - k^2 + \gamma_1k^3)\phi + (\alpha - \gamma_2k)\phi^3 = 0 \tag{82}$$

With reference to Ref. [34], the two Eqs. (81) and (82) can be simplified as follows: Integration Eq. (81) and vanishing the constant of integration, we have

$$\gamma_1\phi'' + (2k - c - 3\gamma_1k^2)\phi + \frac{1}{3}(\gamma_2 + 2\gamma_3)\phi^3 = 0 \tag{83}$$

From Eqs. (82) and (83) we deduce that

$$\frac{\gamma_1}{1-3\gamma_1k} = \frac{2k - c - 3\gamma_1k^2}{\Omega - k^2 + \gamma_1k^3} = \frac{1/3(\gamma_2 + 2\gamma_3)}{\alpha - \gamma_2k} \tag{84}$$

From Eq. (84), we can obtain $k = (\omega - \alpha\gamma_1)/(3\omega\gamma_1 - \gamma_1\gamma_2)$, $\Omega = (1-3\gamma_1k)(2k - c - 3\gamma_1k^2)/\omega + k^2 - \gamma_1k^3$, where $\omega = (1/3)\gamma_2 + (2/3)\gamma_3$. Now, Eq. (84) is transformed into the following form:

$$A\phi'' + B\phi + \omega\phi^3 = 0 \tag{85}$$

where $A = \gamma_1$ and $B = 2k - c - 3\gamma_1k^2$. Balancing ϕ'' with ϕ^3 yields $m = 1$. Consequently, Eq. (85) has the same formal solution of (21). Substituting Eq. (22) and its derivative into Eq. (85) and collecting all term with the same power of $[\exp(-3\varphi(\xi)), \exp(-2\varphi(\xi)), \ldots, \exp(+3\varphi(\xi))]$ we obtained:

$$2A\mu^2a_{-1} + \omega a_{-1}^3 = 0 \tag{86}$$

$$3A\mu\lambda a_{-1} + 3\omega a_{-1}^2a_0 = 0 \tag{87}$$

$$A(2a_{-1}\mu + \lambda^2a_{-1}) + Ba_{-1} + \omega(3a_{-1}^2a_1 + 3a_{-1}a_0^2) = 0 \tag{88}$$

$$A(\lambda a_{-1} + \mu\lambda a_1) + Ba_0 + \omega(6a_{-1}a_0a_1 + a_0^3) = 0 \tag{89}$$

$$A(2\mu a_1 + \lambda^2a_1) + Ba_1 + \omega(3a_{-1}a_1^2 + 3a_0^2a_1) = 0 \tag{90}$$

$$3A\lambda a_1 + 3\omega a_0a_1^2 = 0 \tag{91}$$

$$2Aa_1 + \omega a_1^3 = 0 \tag{92}$$

Solving above system by using maple 16, we get:
Sol. 1.

$$A = \frac{-1}{2}a_1^2\omega, \quad B = \frac{-1}{4}(\omega\lambda^2 - 4\omega\mu)a_1^2$$

$$a_{-1} = 0, \quad a_0 = \frac{1}{2}a_1\lambda, \quad a_1 = a_1$$

Sol. 2.

$$B = \frac{-1}{2}A(4\mu - \lambda^2), \quad a_{-1} = \pm\mu\sqrt{\frac{-2A}{\omega}}$$

$$a_0 = \pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}}, \quad a_1 = 0$$

For Sol. 1.

$$u(\xi) = \frac{a_1\lambda}{2} + a_1\exp(-\varphi(\xi)) \tag{93}$$

Let us discuss the following cases: when $\lambda^2-4\mu>0$, $\mu\neq 0$,

$$u(\xi)=\frac{a_1\lambda}{2}+a_1\exp\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \tag{94}$$

and

$$u(\xi)=\frac{a_1\lambda}{2}+a_1\exp\left(\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}\right) \tag{95}$$

when $\lambda^2-4\mu>0$, $\mu=0$,

$$u(\xi)=\frac{a_1\lambda}{2}+a_1\left(\frac{\lambda}{\exp(\lambda(\xi+C_1))-1}\right) \tag{96}$$

when $\lambda^2-4\mu=0$, $\mu\neq 0$, $\lambda\neq 0$,

$$u(\xi)=\frac{a_1\lambda}{2}+a_1\left(-\frac{\lambda^2(\xi+C_1)}{2(\lambda(\xi+C_1)+2)}\right) \tag{97}$$

when $\lambda^2-4\mu<0$,

$$u(\xi)=\frac{a_1\lambda}{2}+a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \tag{98}$$

and

$$u(\xi)=\frac{a_1\lambda}{2}+a_1\left(\frac{2\mu}{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}\right) \tag{99}$$

For Sol. 2.

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\exp(\varphi(\xi))+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{100}$$

Let us discuss the following cases: when $\lambda^2-4\mu>0$, $\mu\neq 0$,

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\times\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{101}$$

and

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\times\left(\frac{-\sqrt{\lambda^2-4\mu}\coth((\sqrt{\lambda^2-4\mu}/2)(\xi+C_1))-\lambda}{2\mu}\right)+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{102}$$

when $\lambda^2-4\mu>0$, $\mu=0$,

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\left(\frac{\exp(\lambda(\xi+C_1))-1}{\lambda}\right)+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{103}$$

when $\lambda^2-4\mu=0$, $\mu\neq 0$, $\lambda\neq 0$,

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\left(-\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)}\right)+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{104}$$

when $\lambda^2-4\mu<0$,

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\times\left(\frac{\sqrt{4\mu-\lambda^2}\tan((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{105}$$

and

$$u(\xi)=\pm\mu\sqrt{\frac{-2A}{\omega}}\times\left(\frac{\sqrt{4\mu-\lambda^2}\cot((\sqrt{4\mu-\lambda^2}/2)(\xi+C_1))-\lambda}{2\mu}\right)+\pm\frac{\lambda}{2}\sqrt{\frac{-2A}{\omega}} \tag{106}$$

and

4. CONCLUSION

In this research, we introduce a new technique namely the extended $\exp(-\varphi(\xi))$-expansion method for the first time to finding the exact and solitary wave solutions for three different model of equations which play an important role in biology and mathematical physics, these equations are the dynamical system in a new double-Chain model of DNA, the nonlinear Burger equation with power law non linearity and the perturbed nonlinear Schrodinger equation with Kerr law non linearity. From the suggested method we found that it introduce a more accurate and wide range of solutions and these solutions contain the solutions obtained by other methods as tanh method, (G'/G)-expansion method and modified simple equation method. This give more interpretation of the biological and physical properties of the equation studied. We also find that this proposed method is effective, a reliable and a variety of exact solutions NPDEs which can be too applied for many other nonlinear evolution equations.

Competing Interests

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors. The author did not have any competing interests in this research.

Author's Contributions

All parts contained in the research carried out by the researcher through hard work and a review of the various references and contributions in the field of mathematics and the physical applied.

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